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# ON 14-CONNECTED FINITE $H$-SPACES

BY

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ABSTRACT

We prove here that a certain 14 connected finite complex cannot admit the structure of an $H$-space. The $a$ and $c$ invariants of Zabrodsky are used here. It was conjectured by Adams and Wilkerson that the complex described admitted an $H$-structure.

## §0. Introduction

In this note we prove

THEOREM. *There is no* mod 2 *H-space X with*

$$H^*(X; \mathbf{Z}_2) = \mathbf{Z}_2[x_{15}]/x_{15}^4 \otimes \Lambda(x_{23}, x_{27}, x_{29}).$$

This theorem is related to the following theorems:

THEOREM A (Thomas, 1962) [7]. *If X is a finite H-space with primitively generated* mod 2 *cohomology, then the first nonvanishing* mod 2 *cohomology group in degree greater than zero occurs in degree* 1, 3, 7, *or* 15. *Furthermore, if* $H^*(X; \mathbf{Z})$ *is two-torsion free then the first nonvanishing group occurs in degrees* 1, 3, *or* 7.

THEOREM B (Lin, 1987) [4]. *If X is a finite H-space with* $H_*(X; \mathbf{Z}_2)$ *associative, then the conclusion of Theorem A holds. Furthermore, if the first nonvanishing* mod 2 *cohomology occurs in degree* 15, *then there is an* $x_{15} \in H^{15}(X; \mathbf{Z}_2)$ *with* $x_{15}^2 \neq 0$.

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Thus, the theorem of this note eliminates the most elementary case of a finite $H$-space whose first nonvanishing mod 2 cohomology group occurs in degree fifteen. The authors actually make the following conjecture:

CONJECTURE. *Any fourteen-connected finite H-space is acyclic.*

A proof of the theorem of this note using $K$-theoretic techniques has been announced by Ulrich Suter (unpublished). The techniques used in this note are strictly homological. We hope that this note will shed some light on the following problem.

PROBLEM. Let $X$ be a simply connected $H$-space with $H_*(\Omega X; \mathbf{Z}_2)$ finitely generated as an algebra. Classify the algebras that can arise in this way.

In the case of the space $X$ of our theorem above, it is easy to show that

$$H_*(\Omega X; \mathbf{Z}_2) = \Lambda(u_{14}) \otimes \mathbf{Z}_2[u_{22}, u_{26}, u_{28}, u_{58}].$$

The proof of our theorem consists of showing that there is no $H$-space $X$ with $H_*(\Omega X; \mathbf{Z}_2)$ having the above form.

Our argument uses much the same methods as our paper on 6-connected finite $H$-spaces, [6], although the calculations here are different. We wish this paper and [6] to be supplementary, in that this paper will describe the constructions and emphasize the structure of the argument, while [6] contains the full details of calculations.

The basic invariants used in our calculations were described by Zabrodsky in [8]. In particular, we refer the reader to [8] for the definitions of $H$-deviation, $a$-obstruction, and $c$-obstruction.

We wish to thank T. B. Ng for the factorization of $Sq^{32}$ that we use in §1 below.

## §1. First steps

We first note that an $H$-space with the cohomology given above must have the Steenrod connections:

(1.1) $Sq^8 x_{15} = x_{23}$; $Sq^4 x_{23} = x_{27}$; $Sq^2 x_{27} = x_{29}$; and $Sq^1 x_{29} = x_{15}^2$.

Furthermore it follows from [1] that as a co-algebra

$$H^*(\Omega X; \mathbf{Z}_2) \simeq \Lambda(u_{14}) \otimes \Gamma(u_{22}, u_{26}, u_{28}, u_{58}), \tag{1.2}$$

where $u_{14}, \ldots, u_{28}$ are suspensions of the corresponding classes in $H^*(X)$, while $u_{58}$ is the transpotence element associated with the relation $x_{15}^4 = 0$. For the rest of the paper, we set

$$r = u_{14}, \quad s = u_{22}, \quad \text{and} \quad t = u_{26}.$$

We see that the $A(2)$-sub-algebra generated by $u_{14}$ is the algebra

$$B = \mathbf{Z}_2[r]/(r^4) \otimes \Lambda(s, t).$$

We shall construct a stable 3-stage Postnikov system

$$A_2 \xrightarrow{p_2} A_1 \xrightarrow{p_1} A_0$$

such that:

(1) $A_0 = K(\mathbf{Z}_2, 58)$;

(2) there exists an element $v \in PH^{117}(A_2; \mathbf{Z}_2)$ such that $c(v) = (p_1 p_2)^*(\iota_{58}) \otimes (p_1 p_2)^*(\iota_{58})$;

(3) the fiber of $p_2$ is $K(\mathbf{Z}_2; 112, 116) \times K(\mathbf{Z}_2; \text{odd})$ where $K(\mathbf{Z}_2; \text{odd})$ is a product of Eilenberg–MacLane spaces in odd degrees;

(4) the restriction of $v$ to the fiber of $p_2$ is

$$Sq^4 Sq^1 \iota_{112} + Sq^1 \iota_{116} + Sq^{28} \Sigma \alpha_i \iota_{\text{odd}},$$

where $\iota_{\text{odd}}$ is a fundamental class of $K(\mathbf{Z}_2; \text{odd})$ and $\deg(\alpha_i) > 0$.

We note that if it were possible to construct a $c$-map

$$f_2 : \Omega X \to A_2$$

such that

$$f_2^*(p_1 p_2)^*(\iota_{58}) = u_{58},$$

we would obtain an immediate contradiction to the existence of $X$, since

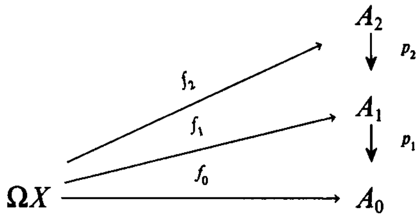
$$\begin{aligned} u_{58} \otimes u_{58} &= f_2^*(p_1 p_2)^*(\iota_{58}) \otimes f_2^*(p_1 p_2)^*(\iota_{58}) \\ &= c(f_2^*(v)) \\ &= 0, \end{aligned}$$

since $f_2^*(v) \in H^{\text{odd}}(\Omega X) = 0$.

But we do not know how to construct such an $f_2$. We are, however, able to contradict the existence of $X$ by constructing maps $f_0$, $f_1$, and $f_2$ that satisfy the hypotheses of the following theorem.

THEOREM 1.1. *If $X$ is an $H$-space, then there is no commutative diagram of $H$-spaces and $H$-maps of the type*

(1.3)




*such that*

(i) $f_0^*(\iota_{58}) = u_{58}$,

(ii) $c(f_1)$ *factors as*

$$\Omega X \wedge \Omega X \xrightarrow{c_1} \Omega^2 K_0 \xrightarrow{\Omega j_0} \Omega A_1$$

*such that*

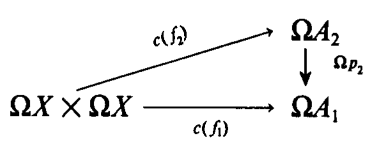
$$\mathrm{Im}(c_1^*) \subset B \otimes B.$$

*(Here $\Omega j_0 : \Omega K_0 \to A_1$ is the fiber of $p_1$.)*

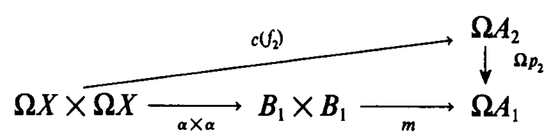
PROOF. By the formula for the $c$-obstruction of a composition [8], we have

$$c(f_2^*(v)) = (f_2^* \otimes f_2^*)(c(v)) + c(f_2)^*(\sigma^*(v))$$
$$= u_{58} \otimes u_{58} + c(f_2)^*(\sigma^*(v)).$$

Since $f_2^*(v) \in H^{\mathrm{odd}}(\Omega X) = 0$, it remains to compute $c(f_2)^*(\sigma^*(v))$. We have a commutative diagram




Condition (ii) implies that there is a map $\alpha : \Omega X \to B_1$, $B_1$ a generalized Eilenberg–MacLane space, such that $\mathrm{Im}(\alpha^*) = B$ and that there is a commutative diagram



Applying the Cartan formula for secondary operations [3, Theorem 3.1], we obtain

$$c_2(f_2)^*(\sigma^*(v)) \in B \otimes H^*(\Omega X) + H^*(\Omega X) \otimes B + \mathrm{Im}(Sq^4 Sq^1) + \mathrm{Im}(Sq^1) + \mathrm{Im}(Sq^{28}\Sigma\alpha_i),$$

where the $\alpha_i$ are elements of positive degree in $\mathscr{A}(2)$. In degrees $\leqq 43$, $H^*(\Omega X) = B$, so

$$\mathrm{Im}(Sq^{28}\Sigma\alpha_i) \subset B \otimes H^*(\Omega X) + H^*(\Omega X) \otimes B,$$

and hence $u_{58} \in B$, a contradiction since $u_{58}$ is indecomposable. □

The remainder of the paper will be devoted to constructing the diagram (1.3). We begin with the definition of the $A_i$'s.

In $\mathscr{A}(2)$, we have the congruence

$$Sq^{60} = Sq^{28} Sq^{32} \mod \mathscr{L},$$

where $\mathscr{L}$ is the left ideal generated by $Sq^{2^k}$, $k = 0, 1, 2, 3$. There is a factorization of $Sq^{32}$ on the intersection of the kernels of $Sq^{2^k}$, $0 \leqq k \leqq 4$:

$$Sq^{32} = Sq^1 \bar{z} + \Sigma\alpha_i\beta_i$$

such that

(1) $\deg \beta_i$ is even,

(2) $\bar{z} = (Sq^{30} + Sq^{26}Sq^4)\phi_{0,0} + (Sq^{28} + Sq^{26}Sq^2 + Sq^{24}Sq^4)\phi_{1,1}$
$+ (Sq^{22}Sq^2 + Sq^{20}Sq^4 + Sq^{18}Sq^6)\phi_{2,2} + Sq^{12}\phi_{2,4}$
$+ (Sq^{16} + Sq^{14}Sq^2 + Sq^{12}Sq^4)\phi_{3,3} + Sq^{18}Sq^4\phi_{1,3}$
$+ Sq^{14}\phi_{1,4} + \phi_{4,4}$,

where $\phi_{ij}$ is the secondary operation based on a factorization of $Sq^{2^i}Sq^{2^j}$.

We now define $A_1$ to be the fiber of

$$g_0 : A_0 = K(\mathbf{Z}_2, 58) \rightarrow K_0 = \prod_{k=0}^{4} K(\mathbf{Z}_2, 58 + 2^k)$$

given by

$$g_0^*(\iota_{58+2^k}) = Sq^{2^k}\iota_{58}, \qquad 0 \leqq k \leqq 4.$$

In $A_1$, we may apply our factorization of $Sq^{32}$ to the image of $\iota_{58}$. Let $z$, $\{b_i\}$ be cohomology classes in $H^*(A_1)$ that represent the operations $\bar{z}$ and $\{\beta_i\}$, respectively. We define $A_2$ to be the fiber of the map

$$g_1 : A_1 \to K(\mathbf{Z}_2, 113) \times K(\mathbf{Z}_2, 117) \times \prod_i K(\mathbf{Z}_2, 58 + \deg \beta_i)$$

given by

$$g_1^*(\iota_{113}) = Sq^{24}z,$$

$$g_1^*(\iota_{117}) = Sq^{26}Sq^2z,$$

and

$$g_1^*(\iota_{58+\deg(\beta_i)}) = b_i.$$

All of the relations used thus far hold stably, so we choose all our spaces and maps to be at least 4-fold loop spaces and maps. If $B^k$ denotes the $k$-fold unlooping functor, we have in $H^*(B^2A_2)$ that

$$\begin{aligned}[(B^2(p_1p_2)^*(\iota_{60})]^2 &= Sq^{60}((B^2p_1p_2)^*(\iota_{60})) \\ &= Sq^{28}Sq^{32}((B^2p_1p_2)^*(\iota_{60})) \\ &= Sq^{28}[Sq^1(B^2p_2^*z) + \Sigma\alpha_i B^2p_2^*(b_i)] \\ &= (Sq^4Sq^1Sq^{24} + Sq^1Sq^{26}Sq^2)(B^2p_2^*z) \\ &\qquad \text{(by the Adem relation } Sq^4Sq^{25} = Sq^{28}Sq^1 + Sq^{27}Sq^2) \\ &= 0.\end{aligned}$$

It follows that there is a suspension element $v \in H^{117}(A_2)$ such that

$$c(v) = (p_1p_2)^*(\iota_{58}) \otimes (p_1p_2)^*(\iota_{58})$$

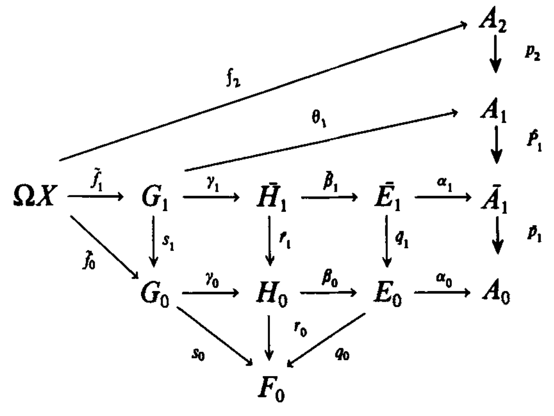
and

$$j_2^*(v) = Sq^4Sq^1\iota_{112} + Sq^1\iota_{116} + Sq^{28}\Sigma\alpha_i\iota_{57+\deg\beta_i}.$$

## §2. Construction of liftings

We now turn to the definition of the maps $f_0, f_1$, and $f_2$ of diagram (1.3). Their construction is somewhat involved; in fact, we need a factorization:

(2.1)




In (2.1) the composition $\bar{p}_1 \hat{p}_1 = p_1$. Here $\bar{p}_1$ is the fiber of the map $g_0'$ given by the same formula as $g_0$ without the $Sq^1\iota_{58}$ term, and $\hat{p}_1$ is the fiber of the map $\bar{g}_0 : \bar{A}_1 \to K(\mathbf{Z}_2, 59)$ given by $\bar{g}_0^*(\iota_{59}) = Sq^1 \bar{p}_1^*(\iota_{58})$.

We shall see that $(\alpha_0\beta_0\gamma_0\tilde{f}_0)^*(\iota_{58}) = u_{58}$, so this composition will serve as our choice of $f_0$. We will choose $f_1 = \theta_1 \tilde{f}_1$. The vertical maps will be principal fibrations whose fibers are generalized Eilenberg–MacLane spaces.

The next theorem specifies the additional properties required in this diagram to produce the hypotheses of Theorem 1.1.

THEOREM 2.1. *The diagram* (2.1), *with the exception of the map* $f_2$, *can be constructed so that*

(a) $G_0$ *is a double loop space*, $H_0$ *and* $E_0$ *are quadruple loop spaces*;

(b) $\alpha_0$ *is a c-map and a-map*, $\beta_0$ *a quadruple loop map*, $\gamma_0$ *and* $\tilde{f}_0$ *loop maps and c-maps*;

(c) $\bar{E}_1$ *is a double loop space*, $\bar{H}_1$ *is the loop space of a c-space*, *and* $G_1$ *is a loop space and c-space*; *the vertical maps respect these structures*;

(d) $\bar{\alpha}_1$ *is an H-map*, $\bar{\beta}_1$ *is a loop map*, $\gamma_1$ *is a loop map*, *and* $\tilde{f}_1$ *is a loop map and c-map*;

(e) $\theta_1$ *is an H-map*.

*Also*

(f) $H^{44}(\bar{E}_1)$ *contains no element whose coproduct is*

$$Sq^8 e_{14} \otimes Sq^8 e_{14}, \qquad e_{14} \neq 0 \in H^{14}(\bar{E}_1):$$

(g) *the fiber of* $\bar{r}_1$ *has its fundamental classes in degrees* $< 72$.

Theorem 2.1 will be proved in Section 3. We now derive the hypotheses of Theorem 1.1 from Theorem 2.1.

THEOREM 2.2. *The obstructions $c(f_1)$ and $a(f_1)$ factor through classes in $B \otimes B$ and $B \otimes B \otimes B$, respectively.*

PROOF. We compute:

$$\begin{aligned} c(f_1) &= c(\theta_1 \tilde{f}_1) \\ &= c(\bar{\theta}_1 \tilde{f}_1) + \text{elements in } H^{57}(\Omega X \wedge \Omega X) && \text{(i)} \\ &= c(\bar{\alpha}_1 \bar{\beta}_1 \gamma_1 \tilde{f}_1) && \text{(ii)} \\ &= c(\bar{\alpha}_1 \bar{\beta}_1 \gamma_1) \circ (\tilde{f}_1 \wedge \tilde{f}_1) && \text{(iii)} \\ &= c(\bar{\alpha}_1 \bar{\beta}_1)(\gamma_1 \tilde{f}_1 \wedge \gamma_1 \tilde{f}_1) + \Omega(\bar{\alpha}_1 \bar{\beta}_1) c(\gamma_1)(\tilde{f}_1 \wedge \tilde{f}_1). \end{aligned}$$

Here (i) holds since $\theta_1$ is a lifting of $\bar{\theta}_1 = \bar{\alpha}_1 \bar{\beta}_1 \gamma_1$, (ii) is true since $H^{\mathrm{odd}}(\Omega X) = 0$, and (iii) holds since $\tilde{f}_1$ is a $c$-map, by Theorem 2.1(d).

By [6, Proposition A5] and the fact that $B\bar{H}_1$ is a $c$-space by Theorem 2.1(c), $(1 + T^*)c(\bar{\alpha}_1 \bar{\beta}_1) = (\Omega^2 g_0)[\eta]$, for some $\eta \in H^{56}(\bar{H}_1 \wedge \bar{H}_1)$. Since $H^{56}(\Omega X) \wedge \Omega X) \subset B \otimes B$, $\Omega^2 g_0$ consists of Steenrod operations, and $B \otimes B$ is closed under these operations, we obtain

$$c(\bar{\alpha}_1 \bar{\beta})(\gamma_1 \tilde{f}_1 \wedge \gamma_1 \tilde{f}_1) \in B \otimes B + \text{diagonal terms}.$$

These terms must lie in

$$H^{29} \otimes H^{29} + H^{30} \otimes H^{30} + H^{32} \otimes H^{32} + H^{36} \otimes H^{36},$$

which is contained in $B \otimes B$, by (1.2).

Application of [5] gives $c(\gamma_1) \in PH^*(G_1) \otimes PH^*(G_1)$. Further, $c(\gamma_1)$ factors through the fiber of $\Omega \bar{r}_1$, so it lies in degrees $\leqq 70$, by Theorem 2.1(g). Hence $\Omega(\bar{\alpha}_1 \bar{\beta}_1) c(\gamma_1)(\tilde{f}_1 \wedge \tilde{f}_1)$ consists of Steenrod operations applied to elements in $\bigoplus_{i,j}(PH^i(\Omega X) \otimes PH^j(\Omega X))$, $i, j \leqq 56$. So

$$\Omega(\bar{\alpha}_1 \bar{\beta}_1) c(\gamma_1)(\tilde{f}_1 \wedge \tilde{f}_1) \in B \otimes B.$$

A similar calculation, using Theorem 2.1(f), yields $a(f_1) \in B \otimes B \otimes B$. (See Theorems 5.1–5.5 of [6] for more detailed versions of these arguments.) □

THEOREM 2.3. *$a(g_1 f_1) \cong *$ and $c(g_1 f_1) \cong *$.*

PROOF. We have

$$a(g_1 f_1) = \Omega g_1 a(f_1) \quad \text{and} \quad c(g_1 f_1) = \Omega g_1 c(f_1).$$

And

$$\Omega g_1 : \Omega A_1 \to K(\mathbf{Z}_2, 112, 116) \times K(\mathbf{Z}_2, \text{odd}),$$

and since $a(f_1)$ and $c(f_1)$ factor through $\Omega^2 K_0$ we see that $a(g_1 f_1)$ and $c(g_1 f_1)$ are the images under Steenrod operations of elements of degrees 58, 60, 64, and 72 in $B \otimes B \otimes B$ and $B \otimes B$, respectively. One calculates, using formula (1.2), that all these images are zero. □

Since $g_1 f_1$ is an $H$-map, it is represented by primitives in $H^*(\Omega X)$, and the degrees of these are $> 58$, $g_1 f_1 \cong *$, so a lifting $f_2$ exists.

THEOREM 2.4. *$f_2$ can be chosen to be an H-map.*

PROOF. We use the method of [8]. $D_{f_2}$ factors through the fiber of $p_2$ which is $K(\mathbf{Z}_2, 112, 116) \times K(\mathbf{Z}_2, \text{odd})$. Hence it factors through a cohomology class $\hat{D}_2$ in degrees 112 and 116. By Theorem 2.3 and [8],

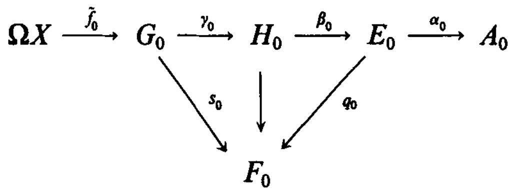
$$(\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})[\hat{D}_2] = [a_3(g_1 f_1)] = 0, \quad \text{and} \quad (1 - T^*)[\hat{D}_2] = [c(g_1 f_1)] = 0.$$

Hence $\hat{D}_2$ determines a class in $P\,\mathrm{Ext}^{2,n}_{H_*(\Omega X)}(\mathbf{Z}_2, \mathbf{Z}_2)$, $n = 112$, and 116. But by (1.2), these groups are zero. Therefore we may alter $f_2$ so that it is an $H$-map. □

Thus the hypotheses of Theorem 1.1 follow from Theorem 2.1.

## §3. Proof of Theorem 2.1

*Step* 1. Construction of



Begin by defining $B^4 q_{-1} : B^4 F_0 \to K(\mathbf{Z}, 18)$ to be the fiber of the map $B^4 w_{-2} : K(\mathbf{Z}, 18) \to K(\mathbf{Z}_2; 20, 22)$ such that

(3.1) $$(B^4 w_{-2})^* \iota_{20} = Sq^2 \iota_{18} \quad \text{and} \quad (B^4 w_{-2})^* \iota_{22} = Sq^4 \iota_{18}.$$

Now let $B^4 E_0$ and $B^4 H_0$ be the fibers of maps $B^4 w_{-1} : B^4 F_0 \to K(\mathbf{Z}_2, 63)$ and $B^4 h_{-1} : B^4 F_0 \to K(\mathbf{Z}_2, 49)$ defined by, respectively,

$$(3.2)\qquad \begin{aligned}(B^4w_{-1})^*(\iota_{63}) &= Sq^{30}Sq^{15}y_{18}\\ &= Sq^{14}Sq^{23}Sq^8y_{18}\end{aligned}$$

and

$$(3.3)\qquad (B^4h_{-1})^*(\iota_{49}) = Sq^{23}Sq^8y_{18},$$

where $y_{18} = B^4w^*_{-2}(\iota_{18})$. We have induced a diagram

$$(3.4)\qquad \begin{array}{ccc} B^4H_0 & \xrightarrow{B^4\beta_0} & B^4E_0\\ \downarrow{\scriptstyle B^4r_0} & & \downarrow{\scriptstyle B^4q_0}\\ B^4F_0 & \xrightarrow{-} & B^4F_0\end{array}$$

Next, define $B^2s_0 : B^2G_0 \to B^2F_0$ to be the fiber of the map $B^2k_{-1} : B^2F_0 \to K(\mathbf{Z}_2, 40)$ given by

$$(3.5)\qquad (B^2k_{-1})^*(\iota_{40}) = Sq^{16}Sq^8y_{16} + y_{16}Sq^8y_{16},$$

where $y_{16} = \sigma^*(\sigma^*y_{18})$. In $H^*(B^2G_0)$, we have the identity

$$(3.6)\qquad \begin{aligned}(B^2s_0)^*(B^2h_{-1})^*(\iota_{47}) &= (B^2s_0)^*(Sq^7Sq^{16}Sq^8y_{16})\\ &= Sq^7(B^2s_0)^*(Sq^{16}Sq^8y_{16})\\ &= Sq^7(y_{16}Sq^8y_{16}).\end{aligned}$$

Looping once, we get

$$(3.7)\qquad \begin{array}{ccc} BG_0 & \xrightarrow{B\gamma_0} & BH_0\\ \downarrow{\scriptstyle Bs_0} & & \downarrow{\scriptstyle Br_0}\\ BF_0 & \xrightarrow{-} & BF_0\\ \downarrow{\scriptstyle Bk_{-1}} & & \downarrow{\scriptstyle Bh_{-1}}\\ K(\mathbf{Z}_2, 39) & \xrightarrow{Sq^7} & K(\mathbf{Z}_2, 46)\end{array}$$

that is commutative and such that $D_{B\gamma_0}$ factors through the map

$$\tilde{D}_{B\gamma_0} : BG_0 \wedge BG_0 \to K(\mathbf{Z}_2, 45)$$

given by

(3.8) $$\tilde{D}_{B\gamma_0}(\iota_{45}) = (Bs_0)^*y_{15} \otimes [(Bs_0)^*y_{15}]^2,$$

by application of a theorem of [9] to (3.6). Consequently, $\gamma_0 : G_0 \to H_0$ is a loop map such that $c(\gamma_0) = 0$.

We next turn to the map $\tilde{f}_0$. Let $P_2X$ denote the projective plane of $X$. Since $x_{15}$ is primitive, it has a representative $P_2x_{15} \in H^{16}(P_2X)$. One checks that

$$Sq^{16}Sq^8(P_2x_{15}) = (Sq^8P_2x_{15})(P_2x_{15}).$$

Also,

$$Sq^2(P_2x_{15}) = 0 = Sq^4(P_2x_{15}).$$

It then follows from (3.1) and (3.5) that the map $P_2X \to K(\mathbf{Z}, 16)$ representing the class $P_2x_{15}$ lifts to a map

$$P_2Bf_0 : P_2X \to B^2G_0,$$

whence arrives an $H$-map

$$Bf_0 : X \to BG_0.$$

Thus $f_0 : \Omega X \to G_0$ is a loop map and $c(f_0) = 0$.

To complete the bottom row of diagram (2.1) we need the map $\alpha_0$. By (3.2),

$$(Bw_{-1})^*(\iota_{60}) = Sq^{30}Sq^{15}y_{15}$$
$$= y_{15}^4.$$

Hence $E_0 \sim F_0 \times K(\mathbf{Z}_2, 58)$, and, by the results of [2], this equivalence is as $a$-spaces. Thus we define the $a$-map $\alpha_0 : E_0 \to A_0 = K(\mathbf{Z}_2, 58)$ to be projection on the second factor. To see that $\alpha_0$ is also a $c$-map, let $B^4\hat{q}_0 : B^4\hat{E}_0 \to K(\mathbf{Z}, 18)$ be the fiber of the map $B^4\hat{w}_{-1} : K(\mathbf{Z}, 18) \to K(\mathbf{Z}_2, 63)$ given by the same formula as $B^4w_{-1}$. As above,

(3.9) $$\hat{E}_0 \sim K(\mathbf{Z}, 14) \times K(\mathbf{Z}_2, 58).$$

Then $\alpha_0$ factors through the projection $\hat{\alpha}_0 : \hat{E}_0 \to K(\mathbf{Z}_2, 58)$. Since $c(\alpha_0) \in H^{57}(E_0 \wedge E_0)$ and the induced map $E_0 \wedge E_0 \to \hat{E}_0 \wedge \hat{E}_0$ is zero on cohomology in degree 57, $c(\alpha_0) = 0$.

*Step* 2. Construction of

$$\begin{array}{ccccccccc}
\Omega X & \xrightarrow{\tilde{f}_1} & G_1 & \xrightarrow{\gamma_1} & \bar{H}_1 & \xrightarrow{\bar{\beta}_1} & \bar{E}_1 & \xrightarrow{\bar{\alpha}_1} & \bar{A}_1 \\
& \searrow^{\tilde{f}_0} & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & G_0 & \xrightarrow{\gamma_0} & H_0 & \xrightarrow{\beta_0} & E_0 & \xrightarrow{\alpha_0} & A_0
\end{array}$$

We now proceed to the second row of diagram (2.1). By (3.3),

$$\begin{aligned}(Bw_{-1})^*(\iota_{46}) &= Sq^{23}(Sq^8 y_{15}) \\ &= (Sq^8 y_{15})^2,\end{aligned}$$

so $H_0 \sim F_0 \times K(\mathbf{Z}_2, 44)$, and $\bar{\Delta}(1 \otimes \iota_{44}) = Sq^8 y_{14} \otimes Sq^8 y_{14}$.

In the next proposition, we collect some information about the images under $\mathscr{A}(2)$ of $1 \otimes \iota_{58} \in H^*(E_0) \cong H^*(F_0 \times K(\mathbf{Z}_2, 58))$ and $1 \otimes \iota_{44} \in H^*(H_0) \cong H^*(F_0 \times K(\mathbf{Z}_2, 44))$.

PROPOSITION 3.1. (1) *There exist elements* $B^2 v_k \in H^*(B^2 E_0)$ *such that* $(\sigma^*)^2(B^2 v_k) = 1 \otimes Sq^{2^k} \iota_{58}$, $1 \leqq k \leqq 4$.

(2) *There are elements* $B^3\alpha_2, B^3\alpha_8 \in H^*(B^3 H_0)$ *and* $B^2\alpha_4 \in H^*(B^2 H_0)$ *that suspend to* $1 \otimes Sq^2 Sq^{14}\iota_{44}$, $1 \otimes Sq^8 Sq^{14}\iota_{44}$, *and* $1 \otimes Sq^4 Sq^{14}\iota_{44}$, *respectively. Furthermore*, $\bar{\Delta}(B^2\alpha_4) = K y_{16}^2 \otimes y_{16}^2$, $K \in \mathbf{Z}_2$.

(3) *There are elements* $B^3\alpha_{16,12}, B^3\alpha_{14,12}, B^3\alpha_{12,1}$, *and* $B^3\alpha_{13}$ *in* $H^*(B^3 H_0)$ *that suspend to* $1 \otimes Sq^{16} Sq^{12}\iota_{44} + y_{14}^2 (Sq^8 y_{14})^2 \otimes 1$, $1 \otimes Sq^{14} Sq^{12}\iota_{44}$, $1 \otimes Sq^{12} Sq^1 \iota_{44}$, *and* $1 \otimes Sq^{13}\iota_{44}$, *respectively.*

(4) $(B\beta_0)^*(Bv_4) = Sq^{17}(B\alpha_{13}) + Sq^{16} Sq^1 B\alpha_{12,11} + Sq^4(B\alpha_{14,12}) + Sq^2(B\alpha_{16,12})$.

These facts arise through appropriate use of the spectral sequence

$$\text{(3.10)} \qquad \operatorname{Tor}_{H^*(Y)}(\mathbf{Z}_2, \mathbf{Z}_2) \Rightarrow H^*(\Omega Y);$$

cf. Propositions 3.1–3.3 of [6], and the Adem relation

$$Sq^{16} Sq^{14} = Sq^{17} Sq^{13} + Sq^{16} Sq^1 Sq^{12} Sq^1 + Sq^4 Sq^{14} Sq^{12} + Sq^2 Sq^{16} Sq^{12}.$$

We choose this particular factorization to ensure part (g) of Theorem 2.1. Details of the proof are left to the reader.

We can now define $\bar{H}_1$, $\bar{E}_1$, and $\bar{A}_1$. Let

$$M_0 = K(\mathbf{Z}_2; 60, 62, 66, 74),$$

and define $B^2 w_0 : B^2 E_0 \to B^2 M_0$ and $B^4 g_0 : B^4 A_0 \to B^4 M_0$ by

$$(B^2w_0)^*(\iota_{60+2^k}) = B^2v_k, \qquad 1 \leqq k \leqq 4,$$

$$(B^4g_0)^*(\iota_{62+2^k}) = Sq^{2^k}\iota_{62}, \qquad 1 \leqq k \leqq 4.$$

Let $B^2\bar{q}_1 : B^2\bar{E}_1 \to B^2E_0$ and $B^4\bar{p}_1 : B^4\bar{A}_1 \to B^4A_0$ be the fibers of $B^2w_0$ and $B^4g_0$, respectively.

Let $B^2L_0 = K(\mathbf{Z}_2; 59, 59, 62, 64, 68, 72, 74)$ and define $B^2h_0 : B^2H_0 \to B^2L_0$ by the table

(3.11)

| $x$ | $(B^2h_0)^*(x)$ |
|---|---|
| $\iota_{59}$ | $B^2\alpha_{12,1}$ |
| $\bar{\iota}_{59}$ | $B^2\alpha_{13}$ |
| $\iota_{62}$ | $B^2\alpha_2$ |
| $\iota_{64}$ | $B^2\alpha_4$ |
| $\iota_{68}$ | $B^2\alpha_8$ |
| $\iota_{72}$ | $B^2\alpha_{14,12}$ |
| $\iota_{74}$ | $B^2\alpha_{16,12}$ |

Let $B^2\bar{r}_1 : B^2\bar{H}_1 \to B^2H_0$ be the fiber of $B^2h_0$.

To define $G_1$, we shall need to deal with $Sq^1$. In $H^{60}(BE_0)$ there is an element $Bv_0$ that suspends to $1 \otimes Sq^1\iota_{58}$ in $H^*(E_0) \approx H^*(F_0 \times K(\mathbf{Z}_2, 58))$. It is not hard to check that

$$\bar{\Delta}(Bv_0) = [(Bq_0)^*(y_{15})]^2 \otimes [(Bq_0)^*(y_{15})]^2.$$

By (2.8), $\bar{\Delta}((B\gamma_0)^*(B\beta_0)^*[Bv_0]) = 0$. We define

$$Bk_0 : BG_0 \to BL_0 \times K(\mathbf{Z}_2, 60)$$

by

$$Bk_0 = Bh_0 B\gamma_0 \qquad \text{on } BL_0,$$

and

$$(Bk_0)^*[\iota_{60}] = (B\gamma_0)^*(B\beta_0)^*[Bv_0].$$

Let $Bs_1 : BG_1 \to BG_0$ be the fiber of $Bk_0$. Since the image of $(B\tilde{f}_0)^*(Bk_0)^*$ consists of primitive elements of $H^*(X)$ in degrees greater than 30, it must be zero. Hence $B\tilde{f}_0$ lifts to $B\tilde{f}_1 : X \to BG_1$. Since $B\tilde{f}_0$ is an $H$-map, $D_{B\tilde{f}_1}$ factors through the fiber of $Bs_1$. Checking degrees, at least one factor of each term of $D_{B\tilde{f}_1}$ is decomposable in $H^*(X)$. Since

$$c(\tilde{f}_1) = (\sigma^* \otimes \sigma^*)(D_{B\tilde{f}_1}),$$

$c(\tilde{f}_1) = 0$. We now have:

(3.12)
$$\begin{array}{ccccccccccc}
 & & & & & & & & A_1 & & \\
 & & & & & & & & \downarrow & & \\
 & & G_1 & \xrightarrow{\gamma_1} & \bar{H}_1 & \xrightarrow{\bar{\beta}_1} & \bar{E}_1 & \xrightarrow{\bar{\alpha}_1} & \bar{A}_1 & \xrightarrow{g_0} & K(\mathbf{Z}_2, 59) \\
 & \tilde{f}_1 \nearrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & \nearrow Sq^1 & \\
\Omega X & \xrightarrow{\tilde{f}_0} & G_0 & \xrightarrow{\gamma_0} & H_0 & \xrightarrow{\beta_0} & E_0 & \xrightarrow{\alpha_0} & A_0 & &
\end{array}$$

By construction, $\bar{A}_1$ is a fourth loop space, $\bar{E}_1$ a double loop space, $\bar{H}_1$ is the loop space of a $c$-space, $G_1$ is the loop space of an $H$-space, and the vertical maps preserve these structures. The maps $\bar{\beta}_1$ and $\gamma_1$ are loop maps, and it is possible to choose $\bar{\alpha}_1$ to be an $H$-map using an argument similar to that of Theorem 2.4. Finally, application of the spectral sequence (3.10) to the space $E_0$ yields part (f) of Theorem 2.1.

It is immediate from the construction that there exists a lifting $\theta_1 : G_1 \to A_1$. It remains to show that $\theta_1$ is an $H$-map. To do this, we begin by obtaining a factorization of the map $Bk_0$. Let $B^2\hat{s}_0 : B^2\hat{G}_0 \to K(\mathbf{Z}, 16)$ be constructed so that, if $B^2g_{14} = B^2\hat{s}_0^*(\iota_{16})$, we have

(i) $Sq^{16}Sq^8(B^2g_{14}) = (B^2g_{14})Sq^8(B^2g_{14}) \neq 0$,

(ii) $Sq^{15}Sq^7(Bg_{14} \otimes Sq^8Bg_{14}) = (Bg_{14})^2 \otimes (Bg_{14})^2 \neq 0$, and

(iii) $PH^{29}(\hat{G}_0) \subset \hat{s}_0^*(H^{29}(K(\mathbf{Z}, 14)))$.

We may also define $B^4\hat{r}_0 : B^4\hat{H}_0 \to K(\mathbf{Z}, 18)$ to be the fiber of the map $B^4\hat{h}_{-1} : K(\mathbf{Z}, 18) \to K(\mathbf{Z}_2, 49)$ given by the formula

$$B^4\hat{h}^*_{-1}(\iota_{44}) = Sq^{23}Sq^8\iota_{18}.$$

There results the commutative diagram

(3.13)
$$\begin{array}{ccccc}
BG_0 & \xrightarrow{B\gamma_0} & BH_0 & \xrightarrow{B\beta_0} & BE_0 \\
\downarrow & & \downarrow & & \downarrow \\
B\hat{G}_0 & \xrightarrow{B\hat{\gamma}_0} & B\hat{H}_0 & \xrightarrow{B\hat{\beta}_0} & B\hat{E}_0
\end{array}$$

By (3.9), there exists $B\hat{v}_0 \in H^{60}(B\hat{E}_0)$ such that $\sigma^* B\hat{v}_0 = 1 \otimes Sq^1 \iota_{58}$. We could have chosen $Bv_0 \in H^{60}(BE_0)$ to be the image of $B\hat{v}_0$, and we assume this choice was made.

Application of the Adem relations to conditions (i) and (ii) gives that $(B\hat{\beta}_0 B\hat{\gamma}_0)^*(B\hat{v}_0)$ is primitive in $H^*(B\hat{G}_0)$. Now let $B\hat{G}_1$ be the fiber of $(B\hat{\beta}_0 B\hat{\gamma}_0)^*(B\hat{v}_0) : B\hat{G}_0 \to K(\mathbf{Z}_2, 60)$. Then $B\hat{G}_1$ is an $H$-space and there is a commutative diagram

(3.14)
$$\begin{array}{ccccc} G_1 & \xrightarrow{s_1} & G_0 & \xrightarrow{s_0} & F_0 \\ \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle q_{-1}} \\ \hat{G}_1 & \xrightarrow{\hat{s}_1} & \hat{G}_0 & \xrightarrow{\hat{s}_0} & K(\mathbf{Z}, 14) \end{array}$$

The $H$-deviation $D_{\theta_1}$ factors through a map $D_1 : G_1 \wedge G_1 \to K(\mathbf{Z}_2, 58)$. By (3.12), $D_1$ can be defined using a nullhomotopy of the composition $Sq^1 \circ \alpha_0 \circ \beta_0 \circ \gamma_0 \circ s_1$. But by (3.9), (3.13), and (3.14), this nullhomotopy could be chosen to be $\delta_1$ composed with a null-homotopy of $Sq^1 \circ \hat{\alpha}_0 \circ \hat{\beta}_0 \circ \hat{\gamma}_0 \circ \hat{s}_1$. Thus the class of $D_1 \in H^*(G_1 \wedge G_1)$ is in $(\delta_1 \wedge \delta_1)^* H^*(\hat{G}_1 \wedge \hat{G}_1)$. By use of the methods of [8] $\theta_1$ can be arranged so that

$$D_1 \in (\delta_1^* \otimes \delta_1^*)(PH^{29}(\hat{G}_1) \otimes PH^{29}(\hat{G}_1)),$$

so by condition (iii), $D_1 = 0$.

Thus Theorem 2.1 is proved.

## References


1. R. Kane, *On loop spaces without p-torsion*, Pac. J. Math. **60** (1975), 189–201.
2. D. Kraines and C. Schochet, *Differentials in the Eilenberg–Moore spectral sequence*, J. Pure Appl. Algebra 2 (1972), 131–148.
3. J. Lin, *Higher order operations in the* mod 2 *cohomology of a finite H-space*, Am. J. Math. **105** (1983), 855–937.
4. J. Lin, *Steenrod connections and connectivity in H-spaces*, Memoirs Am. Math. Soc. **369** (1987), 1–87.
5. J. Lin and F. Williams, *Primitivity of the $c_2$-invariant*, J. Pure Appl. Algebra **43** (1986), 289–298.
6. J. Lin and F. Williams, *On 6-connected finite H-spaces with two-torsion*, Topology, to appear.
7. E. Thomas, *Steenrod squares and H-spaces*, Ann. of Math. **77** (1963), 306–317.
8. A. Zabrodsky, *Cohomology operations and homotopy commutative H-spaces*, in *The Steenrod Algebra and Its Applications*, Lecture Notes in Mathematics #168, Springer-Verlag, Berlin, 1970, pp. 308–317.
9. A. Zabrodsky, *Hopf Spaces*, North-Holland, Amsterdam, 1976.